Team Round

Lexington High School

December 11th, 2021

1. **[30]** Kevin writes the multiples of three from 1 to 100 on the whiteboard. How many digits does he write?

2. **[30]** How many ways are there to permute the letters $\{S, C, R, A, M, B, L, E\}$ without the permutation containing the substring $LAME$?

3. **[30]** Farmer Boso has a busy farm with lots of animals. He tends to $5b$ cows, $5a+7$ chickens, and b^{a-5} insects. Note that each insect has 6 legs. The number of cows is equal to the number of insects. The total number of legs present amongst his animals can be expressed as $\overline{LLL}+1$, where L stands for a digit. Find L.

4. **[35]** Segment AB of length 13 is the diameter of a semicircle. Points C and D are located on the semicircle but not on segment AB. Segments AC and BD both have length 5. Given that the length of CD can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers, find $a+b$.

5. **[35]** In rectangle $ABCD$, $AB = 40$ and $AD = 30$. Let C' be the reflection of C over BD. Find the length of AC'.

6. **[35]** Call a polynomial $p(x)$ with positive integer roots *corrupt* if there exists an integer that cannot be expressed as a sum of (not necessarily positive) multiples of its roots. The polynomial $A(x)$ is monic, corrupt, and has distinct roots. As well, $A(0)$ has 7 positive divisors. Find the least possible value of $|A(1)|$.

7. **[40]** Let $n = 6901$. There are 6732 positive integers less than or equal to n that are also relatively prime to n. Find the sum of the distinct prime factors of n.

8. **[40]** Three distinct positive integers are chosen at random from $\{1,2,3\ldots,12\}$. The probability that no two elements of the set have an absolute difference less than or equal to 2 can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

9. **[40]** Points X and Y on the unit circle centered at $O = (0,0)$ are at $(-1,0)$ and $(0,-1)$ respectively. Points P and Q are on the unit circle such that $\angle PXO = \angle QYO = 30°$. Let Z be the intersection of line XP and line YQ. The area bounded by segment ZP, segment ZQ, and arc PQ can be expressed as $a\pi - b$ where a and b are rational numbers. Find $\frac{1}{ab}$.

10. **[45]** There are 15 people attending math team: 12 students and 3 captains. One of the captains brings 33 identical snacks. A nonnegative number of names (students and/or captains) are written on the NO SNACK LIST. At the end of math team, all students each get n snacks, and all captains get $n+1$ snacks, unless the person's name is written on the board. After everyone's snacks are distributed, there are none left. Find the number of possible integer values of n.

11. **[45]** The LHS Math Team is going to have a Secret Santa event! Nine members are going to participate, and each person must give exactly one gift to a specific recipient so that each person receives exactly one gift. But to make it less boring, no pairs of people can just swap gifts. The number of ways to assign who gives gifts to who in the Secret Santa Exchange with these constraints is N. Find the remainder when N is divided by 1000.

12. **[45]** Let x, y, and z be three not necessarily real numbers that satisfy the following system of equations:

$$x^3 - 4 = (2y+1)^2$$
$$y^3 - 4 = (2z+1)^2$$
$$z^3 - 4 = (2x+1)^2.$$

Find the greatest possible real value of $(x-1)(y-1)(z-1)$.

13. **[50]** Find the sum of

$$\frac{\sigma(n)\cdot \mathrm{d}(n)}{\phi(n)}$$

over all positive n that divide 60.

Note: The function $\mathrm{d}(i)$ outputs the number of divisors of i, $\sigma(i)$ outputs the sum of the factors of i, and $\phi(i)$ outputs the number of positive integers less than or equal to i that are relatively prime to i.

14. **[50]** In a cone with height 3 and base radius 4, let X be a point on the circumference of the base. Let Y be a point on the surface of the cone such that the distance from Y to the vertex of the cone is 2, and Y is diametrically opposite X with respect to the base of the cone. The length of the shortest path across the surface of the cone from X to Y can be expressed as $\sqrt{a+\sqrt{b}}$, where a and b are positive integers. Find $a+b$.

15. **[50]** There are 28 students who have to be separated into two groups such that the number of students in each group is a multiple of 4. The number of ways to split them into the groups can be written as

$$\sum_{k\geq 0} 2^k a_k = a_0 + 2a_1 + 4a_2 + \dots$$

where each a_i is either 0 or 1. Find the value of

$$\sum_{k\geq 0} k a_k = 0 + a_1 + 2a_2 + 3a_3 + \dots.$$